MATHEMATICS AND THE CO-ORDINATION OF MATHEMATICS AND PHYSICS IN SECONDARY SCHOOLS.

THERE is no gainsaying the fact that in New England, at least, the courses of study in secondary schools—and this is especially true in reference to mathematics—are determined largely by the character of the entrance examinations maintained by our leading colleges and scientific schools. This state of things, existing in spite of the fact that only a very small percentage of our boys and girls—figures are dangerous here—ever see the inside of a "higher institution," has arisen in part from the following conditions:

1. The prestige of the college professor in the educational world carries authority with it.

2. Few pupils in a secondary school, whether they really expect to go on or not, like to have the college door swung together in their very faces at this early stage in their educational career, and fewer teachers are willing to assume the responsibility for such action. It follows, therefore, that the course which leads to college must be kept continually in mind by all.

3. Text-books prepared for secondary schools are invariably based upon the college-entrance requirements as interpreted by the aforesaid examinations; for no reliable publisher would dare accept the manuscript of a high-school text that could not be honestly said to cover the ground referred to in the current "requirements," as stated in the catalogues of Harvard, Yale, Dartmouth, and the rest. Witness the lifeless uniformity of the great majority of our elementary text-books both as to subject-matter, order of topics, and method of presentation, as well as the oft-recurring phrase, "designed to cover college requirements."

This being the situation in New England, it will be interesting to look over a few of these college-entrance papers to see if we can discover any important characteristics common to them

all. In Appendix A are found the papers of a dozen of our New England colleges and scientific schools as set for June, 1903. There is evident, at once, a striking uniformity in this, that the problems submitted in algebra are almost without exception of the most abstract character—the factoring of polynominals; the solution of simultaneous quadratic-equations involving x and y; the determination of the coefficient of a given term in an expression to be developed by means of the binominal theorem; the reduction to simple forms of expressions involving fractional and negative exponents; and the summation of series. Rarely a reference to business or to science, or to any feature of daily life. Is it any wonder that our college-preparatory pupils, in their moments of depression, often inquire as to the real use of such a subject as elementary algebra?

A similar condition of things exists in reference to geometry, understood by us all to be the basis of architecture, design, and the various symmetrical forms of nature. The symmetry of the flower, the cell, and the crystal, the proportions of good design, the elements of mechanism, and the salient features of mechanical drawing are made to give way to problems involving incommensurables, limits, extreme and mean ratio, and the computation of π.

The plane trigonometry even presents no hint of a practical application, but runs the weary round of the various transformations occurring in goniometry and the solution of the abstract triangle.

The idea of the examiner appears to be that if the pupil's mathematical tools are in fair condition, it matters not whether he has learned to use them for anything or not.

The various attempts at applied algebra that occasionally appear in these papers are often quite amusing when viewed from the standpoint of daily life. (See Appendix B, 1, 2, 3.)

So much for the nature of the entrance examination—the practical college requirement.

Turning now to the schools themselves, what do we find? Pure mathematics in the saddle: applied mathematics pushed to one side, or neglected altogether. The average pupil upon enter-

ing the high school is able to perform the fundamental operations of arithmetic fairly well, but he has, of course, at this early stage never learned to apply his mathematics to any great extent —especially since so much of mensuration, business arithmetic, and applied problems has been omitted from our grammar-school courses—and he is sadly lacking in logic. He needs training in clearness of thinking and a strengthening of his faith in mathematics as a useful instrument. He usually gets a course in formal algebra, made up largely of "examples"—I use the word advisedly—involving purely abstract quantities, represented generally by x and y, with the addition later of a, b, c, and d, varied occasionally by an applied problem concerning the hands of a watch or the pursuit of a hare by a greyhound. Cross-references to physics, manual training, business affairs, or daily life, or even to the history of mathematics, are almost unknown. The next year comes a course in plane geometry, divorced entirely from any practical application whatever—the dry bones of the subject; a fine training in logic, accurate statement, and keen thinking, I grant, and of inestimable value as such, but lacking the prime essential of practical application. Trigonometry and advanced algebra, treated in a similiar manner, follow.

This state of things cannot last. The unprecedented development of manual training, laboratory methods in science, applied physics based upon mathematics, and our astounding commercial advancement as a nation demand a change. The concrete must go hand in hand with the abstract. "Correlation" must be our watchword. The teachers of mathematics, of natural science, and of manual training must become bosom friends, each vitally interested in and conversant with the work of the other two. Frequent conferences must be held. Points of contact must be eagerly sought after and developed until their number and area are greatly increased. The colleges must help by abridging their requirements in pure mathematics and asking for more applications. In the end, a new subject, known as "everyday mathematics," may be formed by a fusion of physics, manual training, algebra, and geometry; worth infinitely more to the non-college pupil than the present courses can ever be, while form-

ing at the same time a solid basis for subsequent applied work in the college and scientific school.[1]

What will become of the Pure Mathematics, as we know it today, I do not know, but I assume that it will follow the well-known law of the survival of the fittest.

Fusion is in the air. Witness the organization of the Central Association of Science and Mathematics Teachers, with its masterly report of a Committee on the Correlation of Mathematics and Physics in Secondary Schools, to which the writer acknowledges his indebtedness; the formation of the Association of Mathematical Teachers in New England, under the inspiration of Professor Osgood and others; the establishment of an organ, *Supplement to School Science*, devoted to the interests of applied mathematics; and, lastly, the growing sympathy that is manifest everywhere between teachers of pure and applied science. We are on the verge of a new movement. Reforms come slowly. We must not be rash, but each one of us must contribute his share, honestly and fearlessly, in accordance with his best light, to the grand result.

To bring the matter before you, then, not in the form of an ideal course, but as a series of notes in substantial agreement with much of the report referred to, to be discussed and developed, I submit the following points for your consideration:

In our elementary algebra, let us omit or greatly subordinate: (1) H. C. F. by the method of successive division; (2) the factoring of expressions containing two trinomial factors; (3) continued fractions and fractions "many times compounded;" (4) the binomial theorem, except for the square and the cube; (5) cube root; (6) binomial surds; (7) complicated radicals; (8) simultaneous quadratics, excepting those of the simplest character; (9) imaginaries; (10) series; (11) useless puzzles (see Appendix B, 4, 5; Appendix C, 1, 2, 3). Professors D. C. and J. P. Jackson have shown us in their *Elementary Electricity and Magnetism* (Macmillan) how much may be done along practical lines with a very meager mathematical equipment. Let us emphasize

[1] Such a course for beginners may be found in a book entitled *First Course in Mathematics*, published by Doubleday & McClure, New York, N. Y., and edited by SEYMOUR EATON.

the simple equation and its numerous applications—to levers, inclined planes, pulleys, and affairs of daily life; the quadratic as related to the laws of falling bodies and, as far as possible, to the various graphs clustering about the circle—the ellipse, the parabola, and the hyperbola—introducing, in connection with the conic sections, many interesting problems of a simple character concerning centrifugal motion, projectiles, Boyle's law, and planetary motion. Let us teach the practical use of the slide rule, the vernier, and the logarithmic table. Let us make frequent use of formulæ based upon actual data obtained from the laboratory, the workshop, and daily life. These should be manipulated so frequently and so completely that the pupil may come to look upon a formula as a shorthand rule for the solution of a practical problem. The most useful should be memorized and made thoroughly familiar by frequent use and many concrete applications.[1] Let us teach ratio, direct and inverse, as applied to problems in optics, specific gravity, and the like. Let us see that our pupils make frequent use of the graph, discuss its many applications, and carefully consider its interpretations until this form of expression becomes thoroughly familiar. Data for this work can be obtained from railway time-tables, temperature and barometer records as published by the Weather Bureau, observations made in the physical laboratory, and from such books as Haswell's *Mechanic's and Engineer's Pocket-Book*. Let us introduce in great abundance—and much of value can be learned in this connection from the literature of correspondence schools—problems concerning ventilation, heating, and household economics. Our continual aim should be to make the pupil feel that he is getting something that he can use—a new tool.

In geometry, let us agree upon the syllabus published by Harvard University, but rearranged so as to be more nearly in keeping with existing texts. Let us omit limits, incommensurables, and the derivation of π. Let us teach our pupils to treat much of the syllabus as an outline for original demonstration, adding many applications to architecture, mechanical draw-

[1] See SNYDER AND PALMER, *One Thousand Problems in Physics* (Ginn & Co.); PIERCE, *Problems in Elementary Physics* (Henry Holt & Co.).

ing, and physics. Let us display, under the head of "symmetry," a large number of the most beautiful of the geometrical forms of nature, taken from crystallography, botany, and optics, upon the walls of our schoolrooms. Let us review the graph, introduce simple problems from surveying, mechanical drawing, domestic science, and architecture; illustrate mensuration by means of real solids; demonstrate the reality of many theorems by means of squared paper and the balance; review formulæ; use the diagonal scale and the vernier in measurement of lines, and logarithmic tables for computation. Let us teach the nature of errors, and the method of finding the percentage of error. Above all, let us teach constructions as practical men use them. Under parallelograms, let us teach composition and resolution of forces. Under trapezoid, teach simple methods of finding areas. Under construction of triangles with three sides given, let us teach methods of bracing used by bridge-builders and architects; under triangle with one side and two angles given, teach method of locating an inaccessible object, referring by means of a diagram to method of finding the distance to the sun and moon. Under construction of a circle through three points, show shop method of centering. When dealing with the right triangle, let us teach the trigonometric functions, graphically, by means of a simple diagram. The relation of the sides may be illustrated in many ways by paper-folding, and under this head many interesting problems in mechanics may be taken up. Under similar figures, let us teach drawing to scale. Under mensuration of solids, let us review formulæ. Finally the conic sections may be reviewed and a brief introduction to Analytics be made.[1]

Trigonometry should be made intensely practical by use of a simple level and protractor, subordinating goniometry and elevating practice.

Advanced algebra should be reserved for the college.

By way of experiment, the teacher of physics and mathematics might work together, each contributing his share, the recitation being held indifferently in the laboratory, the workshop, or the mathematical recitation room.

[1] T. SUNDARA ROW, *Geometric Exercises in Paper-Folding* (published by Open Court Publishing Co., Chicago); CAMPBELL, *Observational Geometry* (Harper Bros.).

The difficulties in the way of the above reform are many, but they are not insuperable. The college professor will have to give us his opinion as to what can be spared from the present curriculum in preparation for his work, and we shall have to make our substitutions of the concrete and practical for the abstract and theoretical with great care, that we may not mar the symmetry of the pure mathematics, nor emasculate the subject.

To recapitulate:

College-entrance requirements place a premium upon pure mathematics. As a consequence, our secondary schools have taken to emphasizing formal algebra, geometry, and trigonometry to such an extent that cross-references to applied science, and even to the history of mathematics, are almost unknown. The extension of mathematical methods to almost every branch of applied science calls for a change. The sharpening of tools has proceeded far enough. We must begin to acquire skill in their use. Points of contact between physics, manual training, domestic science, and mathematics must be eagerly sought after and developed. The colleges must help by abridging their requirements in pure mathematics and asking for more of the applied. By way of a beginning—not to be too revolutionary—let us omit the least practical portions of the algebra, and add many problems from the departments of applied science, showing clearly the relation between the abstract and the concrete, the theoretical and the practical, let us emphasize the graph and its many applications: expand the subjects of formulæ and ratio, by teaching some of their applications—making use of the physical laboratory in this connection, if need be; let us teach the manipulation of the slide rule, the vernier, caliper, and the logarithmic table; finally let us endeavor constantly to make our pupils feel that algebra is a useful tool as well as an interesting subject for contemplation.

In geometry, let us agree upon the Harvard syllabus, omitting limits, incommensurables, and the derivation of π; let us introduce many problems from architecture, domestic science, mechanical drawing, and physics, making use of the laboratory as far as possible, that the subject may appear more real and the

relation between the abstract and the concrete be more plainly seen; let us make use of squared paper, the diagonal scale, the vernier, and the balance in demonstration of theorems; let us make constant reference to the history of the subject, and thus imbue our pupils with the spirit of some of the old-time heroes of the science; finally, let us strive by every means in our power to impress our pupils with the utility as well as the beauty of the subject; let us treat trigonometry as the handmaid of surveying.

If we do these things, our pupils will at last be able to *do* as well as *see*.

JOHN C. PACKARD.

HIGH SCHOOL,
Brookline, Mass.

APPENDIX A.

Papers referred to were those of (1) Amherst; (2) Bowdoin; (3) Brown; (4) Dartmouth; (5) Harvard, including Lawrence Scientific School; (6) Massachusetts Institute of Technology; (7) Sheffield Scientific School; (8) Smith; (9) Wellesley; (10) Wesleyan; (11) Williams; (12) Yale, Out of 271 problems submitted not over 15 could by any stretch of the imagination be counted as "practical." It is to be remembered that Harvard, to her credit be it said, has long required a course in elementary physics, treated largely as a branch of applied mathematics, for entrance to any department of the university, and sets an examination in the same. This subject is treated by itself, however, and there is but very little attempt at anything like a correlation between pure and applied mathematics apparent.

APPENDIX B.

APPLIED ALGEBRA.

(Harvard, since 1898.)

4. A certain trust company pays interest at the rate of 2 per cent. per annum on the whole of any deposit exceeding $300, and no interest on smaller deposits. A certain savings bank pays interest at the rate of 4 per cent. per annum on all deposits. A man has deposits both in the bank and with the trust company. He transfers $150 from the savings bank to the trust company, and his annual income is increased by two dollars. If, instead of $150, he had transferred a certain different amount from the savings bank to the trust company, his annual income would have been decreased by one dollar. Find the second amount.

The federal inheritance tax on an estate of a dollars is r_1 per cent. of the difference between the value of the estate and the state tax. The state tax is r_2 per cent. of the difference between the value of the estate and the federal tax. Find how much each tax is.

A man rowed every day from A to B against the current of a river, then back with the current from B to A. One day after a heavy rain the river was swollen, and the current was more rapid than usual. The man noticed that he made the return trip from B to A with the current in four-fifths of his usual time for the return trip, but that his time for the whole trip, to B and back, was twice the usual time. How much faster was the current than usual?

Two wheelmen, A and B, are riding eastward over the same road, B's eastward rate exceeding A's by four miles an hour. A certain milestone M is passed by A at noon and by B fifteen minutes later; a second milestone N, six miles east of M, is passed by both wheelmen at the same instant. Find the rate of each wheelman in miles per hour, and the time of their passing N.

5. I travel three hundred miles and back; my return journey being an hour and a half longer than my journey out. If the rate of speed on each journey had been increased by ten miles an hour, the return journey would have been only one hour longer than the journey out. Find the rate of speed on each journey in miles per hour.

The towns A and B are situated on a railway, and also on a highroad running parallel to the railway. At a certain hour a carriage leaves A for B, and, at the same hour, a train leaves B. At three o'clock the carriage and the train are together; the train reaches A at half-past three, and the carriage reaches B at half-past seven. Find the hour at which the carriage leaves A and the train leaves B. (*Two answers.*)

3. Two persons, A and B, run a race to go five times round a certain course. When A has gone three laps, B is 150 yards behind him. A then slackens speed and goes at B's rate, while B quickens his rate and goes at A's first rate. A wins by 30 yards. Find the length of the course, and compare the original speeds of A and B.

1. A policeman is chasing a pickpocket. When the policeman is 80 yards behind him, the pickpocket turns up an alley; but, coming to the end, he finds there is no outlet, turns back, and is caught just as he comes out of the alley. If he had discovered that the alley had no outlet when he had run half way up and had then turned back, the policemen would have had to pursue the thief 120 yards beyond the alley before catching him. How long is the alley?

2. A party of friends went on a pleasure excursion, the expense of which they shared equally. If the number of the party had been decreased by seven, and if the total expense had been \$150, the assessment for each person would have been one dollar more than it was; but if the number of the party had been increased by eight, and if the total expense had been \$160, the assessment for each person would have been one dollar less than it was.

Find the number of the party, the assessment for each person, and the total expense of the excursion.

APPENDIX C.

NEEDLESS COMPLICATIONS.

1. Reduce the following expression:

$$\left[\frac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} \div \frac{1}{a+\cfrac{1}{b}}\right] - \frac{1}{b\,(abc+a+c)},$$

to its simplest form.

2. Prove

$$\frac{x^4+y^4+x^2y^2}{x^3-y^3} \div \frac{x}{x-1}\left\{x-1-\frac{y}{1+\cfrac{1}{x-1}}\right\} = 1+\frac{xy}{(x-y)^2}.$$

3. Solve the equation

$$\frac{\cfrac{c\,(2c-1)}{c+1}x+3}{x} = x + \frac{(c-1)\,(c-2)}{1+c}$$

Solve the equation

$$\left(1-\frac{1}{x}\right)\left(1-\frac{x}{a}\right)-\frac{1}{a}\left(\frac{x}{a}-\frac{2a^2-3x}{x}\right)=0\,;$$

finding two values of x.

Reduce the following expression to a single fraction, having the denominator (a^2-b^2):

$$\frac{2ab}{a+b}-\left(\frac{a}{\sqrt{a+b}}-\frac{b}{\sqrt{a-b}}\right)\left(\frac{b}{\sqrt{a+b}}+\frac{a}{\sqrt{a-b}}\right)$$

4. Write the *fifth* term of $(x-y)^{23}$;

of $(b^{\frac{5}{6}}\sqrt[9]{a^2}-a^{\frac{2}{3}}b^{-2})^{23}$.

Find the twenty-first term of $(x-y)^{24}$. Write, in its simplest form, the value of this term, when

$$x=\sqrt{a^{-\frac{3}{4}}b^{-8}} \qquad\qquad y=(a^{\frac{3}{4}}b^{2})^{\frac{2}{3}}.$$

PROFESSOR EDWIN H. HALL: I believe that I suggested this topic for consideration here and suggested Mr. Packard as the one to deal with it. I feel, therefore, that you, having heard the paper, will agree that I have already made an important contribution to the subject,

and I might perhaps, be excused from taking part in the actual debate. In fact, when I was asked to open the discussion, I did beg to be excused, saying that, as the paper was to be from a teacher of physics, I thought the discussion should be opened by a teacher of mathematics, and I suggested one of my colleagues at college who, I hoped, would be willing to discuss the paper. He declined, and I was asked again, and I found that I could not well avoid taking part.

When I consider the matter I find I have very good warrant for taking part in this discussion, for I find that I am a teacher of mathematics, and every one of my assistants is a teacher of mathematics.

One of my assistants in an elementary course came to me two days ago (he is a laboratory assistant) and asked seriously whether he had not better make some extra appointments to meet the class, to show them about arithmetical processes, how to divide, where to put the decimal point, and how to get the area of a circle from the radius. I said: "No, don't do that, because we have another assistant who is appointed for that special purpose."

Now, is there any way of changing this condition of things? I am not very sanguine that we can do much. I should not like to have pupils who have left my hands, my instruction in physics, taken up a year or two later and questioned upon even the simple things that they had had with me, and have my teaching judged by the accuracy of their answers to simple questions, made on the spur of the moment. One of my mathematical colleagues in Harvard says that a mathematician who has had a reasonably long summer vacation cannot integrate the simplest expression when he gets back to college. I think we shall find in the majority of cases that the boy who cannot divide has at some time known how to divide and how to point off, that by disuse or want of practice for a year or two he has forgotten, and that the simplest hint will set him right, and there will, in this particular, be no great difficulty thereafter.

Nevertheless, the difficulty with mathematics is so real in physics, and in elementary physics, that I think we ought to consider whether there is a possibility of getting help from the mathematicians without impairing the work which they are set particularly to do. I think Mr. Packard has done a service to the mathematicians in calling attention to uses which they could make of material given by the physical laboratory. I wish merely to supplement his suggestions and make them a little more concrete in one or two instances.

In one respect I wish to dissent a little from what I understood to

be Mr. Packard's position, although I may have misunderstood him. I think his suggestions go a little too far in one way, in demanding practical applications for immediate use of everything to be taken up. There is danger in that. If we neglect everything that does not have an obvious use, we should not advance so rapidly as the race has advanced. Nothing seemed less likely to be used a decade or two ago than the glow produced in rarefied gases by an electric current. I do not know of anything that twenty years ago seemed more hopeless, from a point of view of use, than investigations in that field of science. But people kept on, and one day the Roentgen rays were discovered. That paid well for the generations of apparently barren work in the investigation of rarefied gases.

Take one specific recommendation of Mr. Packard's. I think that I should not be in a hurry to give pupils the use of logarithm tables, because it is making them dependent upon a certain machine which they will not always have with them. What we want particularly is to have the pupil capable of thinking out his problems.

Now, let me give examples of the kind of problem, or the kind of formula, I have to use continually with my pupils in teaching physics, and which, it seems to me, might well be utilized by teachers of mathematics. Almost any law that can be expressed in the form of an equation—and there are few laws that cannot be so expressed—can be utilized. Let us take this, the principle of Archimedes:

The weight of a body in liquid is equal to the weight in air[1] *minus the weight of displaced liquid.*

If now we let W' stand for the weight in water, V for the volume of the body, D_1 for the density of the body, and D_2 for the density of the liquid, the principle just stated can be expressed thus:

$$W' = VD_1 - VD_2.$$

Here we have an equation with four quantities, W, V, and the two D's, and we can make use of this equation with a great variety of sets of data. For example, this is a problem that I had with a class yesterday.

A certain body weighs W_1 grams in air. A certain vessel weighs W_2 grams when filled with water. When the given body is placed in the given vessel, displacing a part of the water, the weight of the vessel and its contents is W_3. Find the density of the body.

I found that bothered the class considerably. You see, the trouble there is mathematical. There is no question about this simple prin-

[1] Of course, we here neglect the weight of air displaced by the body.

ciple of Archimedes. The difficulty with the problem, as I say, is all an algebraical difficulty — to see how to take the data given and put them into the form of equations which give the required result; and the difficulty is a perfectly respectable difficulty for a beginner in algebra. I think that would be difficult enough, perhaps, for an algebra paper for admission to Harvard. The whole paper, however, need not be made up of such things.

Take the following as another example:

A tube of uniform cross-section 100 cm. long, closed at one end, is plunged, open end downward, into a mercury-well, and is pushed down until just one half its length is submerged. The tube retains all the air which it held before entering the mercury, and the barometric pressure is 76 cm. How far is the final level of the mercury in the tube below the level of the general mercury surface?

The physical principles which apply in this case are:

1. *Boyle's law, pressure and density, for air.*

2. *In a continuous body of liquid at rest pressure is equally great at all points on the same level.*

3. *Pressure at any depth h in a liquid of density D is equal to the pressure at the top of the liquid plus h D.*

Using these principles, in this problem we get a simple quadratic equation with one unknown quantity. The work is quite difficult enough for an elementary algebra course, and is, in fact, rather too mathematical for an elementary course in physics.

Take problems in heat, in calorimetry. We have this general rule: Heat lost equals heat gained; that is, heat lost by the substances which fall in temperature is equal to heat gained by the other substances. When there are several substances to be considered, each with its proper specific heat, and especially when latent heat has to be reckoned with, we get an equation which looks rather formidable to the average pupil.

Ohm's law in electricity is a law the possibilities of which in elementary teaching are not yet exhausted. I have been giving problems on that for ten or twelve years. It is perfectly easy to make variations.

I of course could give more and more of these, but I think I have gone far enough to give samples.

It seems to me that it would be a service to teachers of mathematics and to teachers of physics alike, if someone would make up a list of these useful principles which can be easily expressed in the form of equations.

As to the teaching of mathematics, by itself, I have one suggestion to make concerning what is commonly given as the first proposition in plane geometry. I went through that proposition with a boy a year ago—a reasonably bright boy, not especially quick in mathematics, but one who gets along fairly well. I want to tell you the impression which that proposition made on the boy and the impression it made on me. The demonstration which I have here copied down is from an old geometry, but it is substantially the same as that in the book which the boy was using.

We have here the figure, a straight horizontal line *ACB*, met by another straight line DC in such a way that the two make an obtuse angle *ACD* and an acute angle *DCB*. A perpendicular *CE* is erected from the point C. With this figure we have the following proposition:

If one straight line meet another straight line, the sum of the two adjacent angles would be equal to two right angles.

Now that is, I think, a perfectly self-evident proposition. Any boy who looks at it, with the figure before him, sees that it is true; and when he is asked to *prove* that it is true, he is in this state of mind: Either the person who asks me to do that is an idiot, and I do not know how to reason with him, or I am an idiot.

Now, this is the demonstration as given. Of course, I don't do it quite justice in reading it off here, but still I will read it slowly:

The angle ACD is the sum of the angles ACE and ECD. Therefore ACD plus BCD is the sum of the three angles ACE, ECD and BCD; but the first of these three angles is a right angle, and the other two make up the right angle ECB. Hence the sum of ACD and DCB is equal to two right angles.

By the time the boy has got through that, I think he is in doubt. Of course this can be justified in a way. We know what geometry is. Geometry is a rigid course of reasoning, where you start with certain agreed-upon material, certain agreed-upon axioms and definitions, and you are to use those only. But I think it is unfortunate to apply that machinery to a problem that is self-evident from the very start. The boy ought to use that machinery at the very outset to prove something which he cannot prove without it. This proposition starts a boy with a discouraged, helpless, feeling, which it takes weeks to get over.

MR. WILLIAM A. FRANCIS, of Phillips Exeter Academy.—Fellow-teachers: I can only say "Me too," as the New York politician said. In truth, I agree most heartily with most that has been said, especially what has been said by the first speaker.

Before I forget it, I will explain, to the best of my ability, the reason why that thing is in geometry. I never wrote a geometry, and I hope I never shall, but I will tell you why it is there. A writer of a text-book thinks first of all, not of the boy, or the teacher, but of the critic. The critic who finds that thing lacking would say: "The book is lacking in rigor." And that is the end of it; you cannot sell the book. I believe the definition of an angle can be so written that the theorem may come under that definition and so omit the difficulty which has been mentioned.

I agree, as I said at the beginning, most heartily with the first speaker, but I wish to take issue with him at the very outset on the matter of advanced algebra from the point of view of the practical, which I think he emphasized a little too much. If anywhere, that is the place to use the practical. Advanced algebra does not all consist of the chance of stealing hens. I think it is possible for the first time in a boy's life to make him understand something of the facts with regard to every great business corporation, not with regard to that problem which was read about shifting money from the savings bank to the trust company, but the facts that lie at the foundation of every sinking fund and the facts that lie at the foundation of insurance. If there is anything in our school mathematics that may be made to touch practical life, it is advanced algebra taught in the right way. We are not all to be bankers, but the matter of insurance in all its forms is worth touching upon. If we are going to look at the practical, I think we must not begin by omitting one of the practical things.

I was converted a long while ago to the belief that we teachers of mathematics must try to teach some physics. The first stage of my conversion came about like this, seven or eight years ago. Harvard, having in mind the old problem about truth in the well, gave a problem in trigonometry about the reflection of an object in the bottom of a well. I had a boy who had done fairly good work, but he put that image in an absurd position, away over on the edge of the well, in a place where no human mortal who had ever studied physics would have put it. I said to him, in reasoning with him afterwards: "You studied physics last year, didn't you?" I never saw a farmer who had bought a gold brick, but I think the expression on that boy's face was something like that. A sort of a cold horror came over him. He said: "Why, they would not put a question in physics in a paper in trigonometry, would they?" That opened my eyes that we must make our teaching cover more practical points. I think we are all agreed that

we must make some definite changes in teaching mathematics. I wish to point out a few dangers in regard to making changes too rapidly, too radically, and too hastily.

The first thing I would warn about is the matter of text-books. I have no reference to text-books in inventional geometry, because I admire Mr. Carpenter's text-book, and other inventional geometries; but I mean in regard to the text-books that will appear in the future, if we make these changes. The publishers are enterprising, and they will set men to work at once to produce certain books, and I am afraid, because no man can make a book without trying it on the boys and girls for a number of years. All the secondary-school books that are worth anything have been made in that way. We must beware of plunging into something new. I have heard mentioned twice in this building within a year a sort of composite book which would teach arithmetic, algebra, geometry, and trigonometry, all in one book. The speaker last evening, I think, suggested it. I hope it will be made; I hope he will make it; but I hope he will not make it until he spends five years in experimental use on boys and girls. I do not know how to make such a book. I hope some of you do, and I hope if you do you will make one.

With regard to these hasty ways that sweep over the country, in mathematical teaching, one of them was one that appeared a few years ago on heuristic geometry. We do not hear so much about that now. The enthusiasts thought they had solved the trouble with the geometry of the secondary schools by making an entirely new invention. I went to visit a school on purpose—in a friendly spirit, to be sure—to see how the thing worked, with one of the most enthusiastic teachers in New England. He claimed it was wrong that the children should take their ideas second-hand—that they should find them all for themselves. I think I state the matter fairly when I say that of a class of eighteen, ten boys and eight girls, there were four boys and two girls doing all the thinking, and the other twelve of the eighteen took their ideas entirely second-hand from the six. I watched the faces of the slower members of the class when a new question arose. They never thought of thinking about the thing, but waited to see what the others thought. I think that is a fair representation of the thing.

It seems to me that, in regard to highest common factor by division, the outcry against it is due to the fact that the problem is made too hard. It is put in a place in the algebra where it tests a boy's ability over the elementary processes—whether he can add, subtract, multiply,

and divide. And another thing, if we wish to bring these different subjects closer together, boys hear the fact that the greatest common divisor and highest common factor and finding the ratio of two commensurable straight lines is exactly the same process. For that purpose alone I should like to keep it, although some institutions, like the Massachusetts Institute of Technology, have hinted that perhaps it would be better to omit it in elementary training. There are other things, like the gymnastics that were given by the first speaker, that I should like to see omitted in the school curriculum. There are some things that we may omit from our geometry too, as maxima and minima and the theory of limits. But there is one thing I should like to plead for here, that is, for the boy who can understand the theory of limits, and there are such boys. I think we should realize in the teaching of secondary schools that we may spoil mathematicians. I spoil some every year. I am sorry for it, but I am sure I do by not giving them enough to do to occupy their minds. Dividing my class roughly, I think about a third of the class understands the theory of limits. I think that third should have the right and chance to study that matter. The other two-thirds are very much like the boy who wrote this in good faith five or six years ago: "Case II, when the arcs are incomprehensible."

There is one word that has been used here a great deal today that I am very much afraid of. It is that word "practical." If we were to point to the subject which is worst taught in the elementary schools, I think all teachers, in mathematics at least, would say it is arithmetic. As the last speaker said, it is unfair to judge a teacher by the work he has done two or three years before, and yet this is my experience in taking up a class in logarithms. I spend several hours in trying, before they begin, to teach them to use a table so that it shall be an instrument to save time and to aid accuracy. In the first written test I say to them: "You write on this page everything which I am to read. On the opposite page put your scribbling work for interpolation, and put just as little as possible on that opposite page." There is a good deal more on the opposite page than on the page I am to read. It is pitiful to find boys within one year of college who dare not trust themselves to get $\frac{6}{10}$ of 13. You say: "Why don't you teach them better?" We try to, and we come to sympathize with the position of the physician of Lady Macbeth.

If there is one thing that the school should enable the boy to do, it is to multiply numbers no harder than 7 times 16 and have confi-

dence that he is likely to be right sometimes. If the rest of you have had a different experience from that, it is because your boys are drawn from parts of the country, perhaps, in which the teaching is good. I get boys from every state in the Union, and it is pitiful to see their struggles with a table of logarithms. I have had four hours of it this week, so you will pardon my reference to my suffering.

In making substitution for the geometry which we teach now, I have no agreement with the gentleman who spoke last evening, that there is very little educational value in demonstrational geometry, and I feel sure that most of you teachers in geometry have not. We must bear in mind this thing, that when we substitute various things, like mechanical drawing and so on, for that kind of geometry, we substitute a thing which is by no means equal in the present stages of the work. It may be equal one of these days, but it is not now. Some of the most hopeless boys I have in geometry are boys who have had two years, perhaps, in mechanical drawing in various schools, and have done the drawing well, but it has given them no idea of geometrical forms.

The first speaker said—and I agree with him—that the colleges must help us. And there is still a great chance for them to help us in one matter that has not been mentioned today, I think. I know that the Harvard papers have been improved very much in elementary mathematics since that problem, when A rode north at six miles an hour and B rode south at a fixed rate—I don't remember the rest of it. It is only a few years ago we were asked to prove that two lines which were perpendicular were parallel, and we were asked to discuss a spherical triangle, which was not a spherical triangle in the ordinary sense in which we use the term. I feel that examination papers are not carefully made with a view to considering the effects upon the boy.

I have one grievance in mind, and that is the September paper of this year in advanced algebra for Harvard College. Time given, one hour. The first question was a question in continued fractions; the second, an arithmetical progression; the third, a question in chance; the fourth, a double question in determinants; the fifth, two questions on the theory of equations; the sixth, a question to approximate a root to three decimal places. That paper represents what happened when they added to our work the teaching of determinants. Just 20 per cent. was added to the difficulty of the paper, and not one minute more of time given.

I spend a great deal of my time in June, when life is a burden and

digestion fails, in putting the boys through a training just exactly like the football training, in order to teach a boy not to run, but to sprint, that he may do more in an hour than he ought to try to do. It is useless to say to the boy: "They do not expect you do six questions, but five." The boy never lives up to that idea. He does two questions, and he sees a half-hour is gone, and then he works in a way that no human being can use and do good work. His work is hurried, often unreasonable, and sometimes entirely senseless. The reason is that in a paper like that he is asked to do in one hour work for which most colleges in a like course give an hour and a half. We could do a great deal more effective work in fitting our boys for the examination if we were not obliged to put them through those gymnastics just simply to make speed.

It is well enough to review. We must review. A boy who knows the quadratic equation does not know it three months from today if he has not used it.

It is natural that teachers of mathematics are dissatisfied with their work; we ought to be. I hope we are all going to try along those lines laid out by the first speaker to make some improvements. The first place to begin is in the work of inventional geometry. The new suggestion, to combine physics and mathematics so as to use effectively the problem that will appeal to the student, is a thing we can all take to heart, but we must be on our guard all the time to remember that when we substitute mere manual training for logical thinking, we are putting in something which is easier, and which can by no means entirely replace the other; and we must remember that, while algebra and geometry which we teach will pass, the habit of thinking straight, which most of our boys and girls may attain, may possibly remain.